# Math 2050, summary of Week 11

## 1. Uniform continuity

Recall that $f : A \to \mathbb{R}$ is said to be uniform continuous if $\forall \varepsilon > 0$, there is $\delta > 0$, such that for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

In contrast with continuity of a function, *Uniform continuity is a global properties*!

**Question:** How to improve from continuity to uniform continuity?

**Theorem 1.1.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is uniformly continuous.*

We here present an alternative proof different from that in the textbook. (more complicated but more intuitive )

*Proof.* Let $\varepsilon > 0$ be given. We consider the following subset $S$ of $[a, b]$:

$$S = \{c \in [a, b] : \ \exists \delta > 0, \text{ such that } \forall x, y \in [a, c], |x-y| < \delta, \Longrightarrow |f(x)-f(y)| < \varepsilon\}.$$

Clearly, $a \in S$ and $S$ is bounded from above. Hence, $s = \sup S \leq b$ exists. We will show that $s = b \in S$. Suppose not, $s < b$. By continuity of $f$, there is $\delta_s > 0$ such that for all $x \in [a, b]$ and $|x - c| < \delta_s$, we have

$$|f(x) - f(c)| < \frac{1}{2}\varepsilon.$$

On the other hand, since $s - \frac{1}{2}\delta_s < s$, there is $c \in S$ such that $s - \frac{1}{2}\delta_s \leq c$ and hence, we can find $\delta_c > 0$ so that for all $x, y \in [a, c]$ with $|x - y| < \delta_c$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Now we choose $\delta = \min\{\delta_c, \frac{1}{4}\delta_s\} > 0$. For $x, y \in [a, s + \frac{1}{4}\delta_s] \cap [a, b]$ with $|x - y| < \delta$.

If $x, y \leq s - \frac{1}{2}\delta_s$, then we must have $|f(x) - f(y)| < \varepsilon$ since $|x - y| < \delta_c$. If $x \leq s - \frac{1}{2}\delta_s < y$, we have $|x - s|, |y - s| < \delta_s$ since $|x - y| < \frac{1}{4}\delta_s$, therefore

$$|f(x) - f(y)| \leq |f(x) - f(c)| + |f(y) - f(c)| < \varepsilon. \tag{1.1}$$

If $s - \frac{1}{2}\delta_s < x, y < s + \frac{1}{2}\delta_s \leq b$, then the same argument shows that $|f(x) - f(y)| < \varepsilon$. Hence, $s + \frac{1}{2}\delta_s \in S$ which is impossible. This shows that $b = s$. Moreover, the same argument shows that $s \in S$. This completes the proof. ∎

From the Theorem, the continuity on closed and bounded interval is automatically uniform continuous. The inverse is also true in a suitable sense.

**Theorem 1.2.** *Suppose* $f : (a, b) \to \mathbb{R}$ *is uniformly continuous, then there is* $\tilde{f} : [a, b] \to \mathbb{R}$ *such that* $\tilde{f} = f$ *on* $(a, b)$ *and is continuous on* $[a, b]$.

*Proof.* It suffices to show that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist. Let $x_n \to a^+$. Since a uniformly continuous function on bounded interval must be bounded, $\{f(x_n)\}$ is a bounded sequence. Therefore, there is $x_{n_k} \to a$ such that $f(x_{n_k}) \to L \in \mathbb{R}$ for some $L$.

Suppose $\lim_{x\to a^+} f(x)$ doesn't exists. Then for $L \in \mathbb{R}$, we can find $y_n \to a^+$ such that $f(y_n) \to l \in \mathbb{R}$ but $l \neq L$. But this contradicts with the uniform continuity as both $x_n$ and $y_n$ converges to $a$.

□